

# Amplification induced by a periodic coefficient and a multiplicative white noise term

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The author studied the amplification of the the amplitude in a Mathieu-like equation with multiplicative white noise. An expression of the exponent on parametrically amplified regions was derived theoretically by introducing the width of time interval, and the exponents were calculated numerically by solving the stochastic differential equations by a symplectic numerical method. It was shown that the exponent decreases with a parameter  $\alpha$ , reaches the minimum and increases after that, where the value of  $\alpha$  is determined by the intensity of noise and the strength of the coupling between the variable and the noise. The author found that the exponent as a function of  $\alpha$  has only one minimum at  $\alpha \neq 0$  on parametrically amplified regions of  $\alpha = 0$ . This minimum value is obtained theoretically and numerically. The existence of the minimum at  $\alpha \neq 0$  indicates the suppression of the amplification by white noise.

KEYWORDS: Parametric Resonance, Gaussian White Noise, Exponent, Suppression

## 1. Introduction

In past few decades, many researchers have investigated the roles of noise, and then remarkable phenomena were found. For examples, such phenomena are stochastic resonance,<sup>1-4)</sup> phase transition induced by multiplicative noise<sup>5)</sup> and so on.<sup>6-10)</sup> A basic system in which multiplicative noise acts is an oscillator with varying mass and/or friction. The amplification by noise is important, because it brings the unstableness of the state, the clear signal and so on.

The role of noise acting on a harmonic oscillator multiplicatively was investigated by Stratonovich.<sup>11)</sup> He found that the amplitude of the oscillator with friction is amplified by noise. Recently, the phase transition was investigated for pendulum with a randomly vibrating suspension axis.<sup>12)</sup> Mallick and Marcq investigated nonlinear oscillators with white or colored noise.<sup>13-15)</sup> They showed that the time dependence of the amplification obeys power law.

Another mechanism of the amplification is parametric resonance.<sup>16)</sup> Some physicists have already been investigated the effects of additive white noise acting on the harmonic oscillator with a periodic coefficient.<sup>17)</sup> They showed that the covariance of a coordinate has a minimum value at a nonzero value of the parameter which is included in the periodic coefficient. In the study of a parametric oscillator driven by a periodic coefficient with additive white noise,<sup>18,19)</sup> the suppression of the fluctuation was

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found. The parametric resonance induced by multiplicative colored noise was investigated in ref. 20. In field theories, it was shown that a field is amplified by another oscillating field. The presence of noise leads to an increase in the rate of particle production.<sup>21)</sup> In  $\phi^4$  theory, the equation of motion of the field of a nonzero mode near the bottom of the effective potential may have an oscillating mass term with noise.<sup>22)</sup> In such a system, the magnitude of the amplification is modified by white noise quantitatively. A Mathieu-like equation which has a multiplicative noise term may appear even in the end of phase transition, as shown in ref. 22. Experimentally, some physical systems which are described by the equations with a periodic coefficient and a multiplicative noise term were studied.<sup>23)</sup> The equations to describe such systems are similar to the equation of pendulum.

The differential equation with a periodic coefficient and a multiplicative noise term appears in many physical systems. Multiplicative noise may amplify or suppress the amplitude, as found in the case of additive noise. The effects of multiplicative white noise on the system which shows the parametric amplification should be investigated, because the suppression of the amplitude is directly related to the stability of the system.

In this paper, I studied the effects of multiplicative white noise acting on the system that the parametric resonance occurs when the values of the parameters of the system are adequate. A stochastic differential equation was analyzed by introducing the width of time interval, and I derived an approximate expression of the exponent on parametrically amplified regions where parametric resonance occurs when no noise exists. The behavior of the exponent was discussed qualitatively and the minimum value of the exponent as a function of the parameter  $\alpha$  was estimated, where this parameter indicates the intensity of noise and the strength of the coupling between the variable and the noise. The stochastic differential equations were solved numerically by a symplectic method to avoid the amplification by numerical error, and the exponent was extracted from the average of the trajectories. The behavior of the exponent as a function of  $\alpha$  was displayed numerically with the other parameters fixed. I found that the exponent has only one minimum at  $\alpha \neq 0$  on parametrically amplified regions of  $\alpha = 0$ . The existence of the minimum indicates the suppression of the amplification by white noise. The results provide insight in the systems with periodically varying parameters and multiplicative noise.

## 2. Qualitative properties of the exponent on parametric amplified region

### 2.1 The basic equation

The equations with a periodic coefficient and a multiplicative white noise term are interested in many branches of physics. We come up with the following equation which describes some phenomena (under some conditions), for examples, the motion of surface wave,<sup>23)</sup> the time variation of the charge in a capacitor,<sup>23)</sup> reheating in inflationary cosmology,<sup>21)</sup> motion of the condensate near the potential

well<sup>22)</sup> and so on:

$$\frac{d^2\phi}{dt^2} + [\omega^2 + B \cos(\Omega t + \theta) + Dn(t)]\phi = 0, \quad (1)$$

where  $\omega$ ,  $B$ ,  $D$ ,  $\Omega$  and  $\theta$  are constants, and  $t$  is time. The quantity  $n(t)$  is white noise with the property,  $\langle n(t) \rangle = 0$  and  $\langle n(t)n(t') \rangle = \delta(t-t')$ , where the notation  $\langle \dots \rangle$  represents statistical average. By applying the transformation  $z = \omega t + \theta\omega/\Omega$ , this equation is transformed to

$$\frac{d^2\phi}{dz^2} + [1 + \beta \cos(\gamma z) + \alpha r(z)]\phi = 0, \quad (2)$$

where the quantities  $\alpha$ ,  $\beta$  and  $\gamma$  are defined as follows:

$$\alpha = D/\omega^{3/2}, \quad \beta = B/\omega^2, \quad \gamma = \Omega/\omega. \quad (3)$$

The quantity  $r(z)$  is defined by  $r(z) = \omega^{-1/2}n(z/\omega - \theta/\Omega)$ . Therefore  $r(z)$  has the following properties:

$$\langle r(z) \rangle = 0, \quad \langle r(z)r(z') \rangle = \delta(z - z'). \quad (4)$$

The starting point in this paper is eq. (2) with eq. (4).

Equation (2) is rewritten with the variable  $p_\phi$  which is defined by  $p_\phi = d\phi/dz$ :

$$d\phi = p_\phi dz, \quad (5a)$$

$$dp_\phi = -[1 + \beta \cos(\gamma z)]\phi dz - \alpha \phi \circ dW. \quad (5b)$$

The quantity  $W(z)$  is defined by  $W(z) = \int_{z_0}^z ds r(s)$  and this is a wiener process, where the quantity  $z_0$  is an initial time. (Here, the symbol  $\circ$  represents Stratonovich product.) I attempt to solve eqs. (5a) and (5b) numerically in § 3.

Equation (2) is just a Mathieu equation when  $\alpha$  is zero, and this equation has resonance bands. The Mathieu equation corresponding to eq. (2) with the relation  $2u = \gamma z$  is given by

$$\frac{d^2\phi}{du^2} + (a - 2q \cos(2u))\phi = 0, \quad (6)$$

where  $a = 4/\gamma^2$  and  $-2q = 4\beta/\gamma^2$ . Then the bands are distinguished by positive integer  $n$  with the relation  $n^2 = 4/\gamma^2$ . Therefore the values of  $\gamma$  in resonance bands are close to  $2/n$  at  $\alpha = 0$ .

## 2.2 Minimum Value of the Exponent on parametrically amplified regions

In this subsection, I attempt to estimate the growth rate of the field amplitude in time. I call this rate the exponent in this paper. This rate is given by  $\limsup z^{-1} \ln [|\langle \phi(z) \rangle| / |\phi(0)|]$ , where  $\phi(0)$  is the initial value. I use the solution of the Mathieu equation in order to solve eq. (2) approximately in the region where parametric resonance occurs at  $\alpha = 0$ . The equation at  $\alpha = 0$  is

$$\ddot{\Phi} + [1 + \beta \cos(\gamma z)]\Phi = 0, \quad (7)$$

where the dot represents the derivative with respect to  $z$ . The quantity  $\phi$  is represented as a product of  $\Phi$  multiplied by a new variable  $\psi$ :

$$\phi = \Phi\psi. \quad (8)$$

By substituting eq. (8) into eq. (2),  $\psi$  satisfies the subsequent equation:

$$\ddot{\psi} + 2(\dot{\Phi}/\Phi)\dot{\psi} + \alpha r(z)\psi = 0. \quad (9)$$

The exponent of  $\Phi$  was investigated by many researchers in detail. Therefore, the exponent of  $\phi$  is estimated by obtaining the exponent of  $\psi$  approximately.

Here I denote the exponent of  $\phi$  at  $\alpha = 0$  as  $s \equiv s(\beta, \gamma)$  which is just the exponent of  $\Phi$ . The time dependence of  $\Phi$  is obtained by solving eq. (7). One method to solve approximately in the first resonance band is performed by putting the form of  $\Phi$  with the assumption  $\ddot{P} \sim 0$  as follows:<sup>16,24–26)</sup>

$$\Phi = P(z)e^{i\gamma z/2} + P^*(z)e^{-i\gamma z/2}. \quad (10)$$

This solution gives the approximate expression of  $\Phi$ . From this calculation, I obtain

$$\Phi \sim e^{s_1 z} F_1(z), \quad (11a)$$

$$F_1(z) := C e^{i\gamma z/2} + C^* e^{-i\gamma z/2}, \quad (11b)$$

where  $C$  is a complex constant and  $s_1$  is the exponent. It is conjectured that the exponent  $s_1$  is close to the exponent  $s$  in the first resonance band. With eqs. (11a) and (11b), I obtain

$$\dot{\Phi}/\Phi \sim s_1 + \dot{F}_1/F_1. \quad (12)$$

The exponent is estimated by solving eq. (9) with eq. (12). However, it is not easy to handle eq. (9). Instead, in eq. (9), I replace  $(\dot{\Phi}/\Phi)$  by the average of  $(\dot{\Phi}/\Phi)$  in time. This replacement is probably valid when the change of  $\psi$  is slow enough. The condition that this replacement is valid is extracted as follows. The equation for  $\psi$  is described as

$$\frac{d}{dz} \left[ (F_1)^2 \frac{d}{dz} \psi \right] + 2s_1 \left[ (F_1)^2 \frac{d}{dz} \psi \right] + \alpha r(z) (F_1)^2 \psi = 0. \quad (13)$$

The solution for small  $\alpha$  will be close to the solution at  $\alpha = 0$  which satisfies  $(F_1)^2 \dot{\psi} = C \exp(-2s_1 z)$ , where  $C$  is constant. Here I denote the period of  $F_1(z)$  as  $T$  and the time scale of the exponential decrease as  $T^*$ . The variation of  $\psi$  is slow when the absolute value of  $\dot{\psi}$  is small. It is obvious from this approximate solution,  $(F_1)^2 \dot{\psi} = C \exp(-2s_1 z)$ , that the absolute value of  $\dot{\psi}$  is small when the function  $F_1(z)$  is approximately constant in the width  $T^*$ . Therefore the solution obtained by this replacement will be valid when the condition,  $T \gg T^*$ , is satisfied. I put  $F_1(z) = \cos(\gamma z/2)$  in order to check this condition. In this case,  $T$  is  $4\pi\gamma^{-1}$ . Then this condition is expressed by the inequality  $8\pi s \gg \gamma$ , because  $T^*$  is estimated as  $1/(2s)$ . The value of  $\gamma$  on the first resonance band is close to 2 and the value of  $s_1$  is

about 0.45 at  $\beta = 2$  and  $\gamma = 2$ . Then this inequality holds at  $\alpha = 0$ . Therefore the replacement by the average should be available to extract the properties of the exponent for small  $\alpha$ . Keeping these facts in mind, I estimate the averaged value of the exponent  $\psi$ . The average of  $\dot{\Phi}/\Phi$  in one period of  $F_1(z)$  is equal to  $s_1$ . Therefore, the approximate equation for  $\psi$  under this approximation in the first resonance band is

$$\ddot{\psi} + 2s_1\dot{\psi} + \alpha r(z)\psi = 0. \quad (14)$$

I derive an approximate equation in other resonance bands in the similar way. The variable  $\Phi$  is expanded as

$$\Phi = \sum_{n=1} \left[ P_n(z) e^{inyz/2} + P_n^*(z) e^{-inyz/2} \right] + R(z). \quad (15)$$

The growth of the function  $P_m(z)$  is largest in  $m$ th resonance bands. Therefore,  $\Phi$  in the  $m$ th band is approximately given by

$$\Phi \sim e^{s_m z} F_m(z), \quad F_m(z) := C e^{imy z/2} + C^* e^{-imy z/2}, \quad (16)$$

where  $s_m$  is the exponent. The equation for  $\psi$  becomes

$$\ddot{\psi} + 2s_m\dot{\psi} + \alpha r(z)\psi = 0. \quad (17)$$

Then I attempt to solve the subsequent equation for  $\psi$  in the resonance band:

$$\ddot{\psi} + 2s\dot{\psi} + \alpha r(z)\psi = 0. \quad (18)$$

The purpose in this paper is to estimate the exponent. Then I put the form of  $\psi$  as follows:

$$\psi = \psi_0 \exp \left( \int_{z_0}^z dz' \sigma(z') \right). \quad (19)$$

Substituting eq. (19) into eq. (18), I obtain the equation for  $\sigma$ :

$$\dot{\sigma} + \sigma^2 + 2s\sigma + \alpha r(z) = 0. \quad (20)$$

At first, I find the solution when  $r(z)$  is constant. The solution of eq. (20) is categorized by the quantity  $\mathcal{D}$  which is defined as  $4s^2 - 4\alpha r$ . For  $\mathcal{D} > 0$ , I have

$$\int_{z_0}^z dz' \sigma(z') = \left( -s + \sqrt{\mathcal{D}}/2 \right) (z - z_0) + \frac{1}{\sqrt{\mathcal{D}}} \ln \left| \frac{1 - C e^{-\sqrt{\mathcal{D}}z}}{1 - C e^{-\sqrt{\mathcal{D}}z_0}} \right|, \quad (21)$$

where  $C$  is an arbitrary constant. For  $\sqrt{\mathcal{D}}z \gg 1$ , the last term of the right-hand side in eq. (21) is negligible for the exponent. For  $\mathcal{D} < 0$ , I have

$$\int_{z_0}^z dz' \sigma(z') = -s(z - z_0) + \ln \left| \frac{\cos \left( \frac{\sqrt{-\mathcal{D}}}{2} z_0 + C' \right)}{\cos \left( \frac{\sqrt{-\mathcal{D}}}{2} z + C' \right)} \right|, \quad (22)$$

where  $C'$  is constant. The second term in the right-hand side of eq. (22) does not contribute to the

amplification, though it enhances and suppresses the amplitude temporarily. For  $\mathcal{D} = 0$ , I have

$$\int_{z_0}^z dz' \sigma(z') = -s(z - z_0) + \ln \left| \frac{z + C''}{z_0 + C''} \right|, \quad (23)$$

where  $C''$  is constant. The second term in the right-hand side of eq. (23) does not contribute to the amplification substantially.

Next, I consider the case that the quantity  $r(z)$  is time dependent. For such the case, the region  $[z_0, z]$  is divided into small regions with the width of time interval  $\Delta z$ . Moreover, the region with the width  $\Delta z$  is divided into quite small  $N$  regions numbered ' $j$ ' in which the quantity  $r$  is constant. I define the quantity  $\Delta W_j$  by  $r_j \Delta z / N$ , where  $r_j$  is the value of  $r$  in the region ' $j$ '. This quantity  $\Delta W_j$  is a wiener process and the distribution function of  $\Delta W_j$  is given by

$$P(\Delta W_j) = \frac{1}{\sqrt{2\pi(\Delta z)/N}} \exp \left( -\frac{(\Delta W_j)^2}{2(\Delta z)/N} \right). \quad (24)$$

Then the quantity  $\Delta W \equiv \sum_{j=1}^N \Delta W_j$  obeys the distribution function  $P(\Delta W)$  which is given by

$$P(\Delta W) = \frac{1}{\sqrt{2\pi(\Delta z)}} \exp \left( -\frac{(\Delta W)^2}{2(\Delta z)} \right). \quad (25)$$

Therefore, the values of  $\Delta W$  in the regions with the width  $\Delta z$  are distributed with the probability  $P(\Delta W)$ . Then the statistical average of a variable  $O$  is taken as follows:

$$\langle O \rangle = \int_{-\infty}^{\infty} d(\Delta W) P(\Delta W) O. \quad (26)$$

From eqs. (16), (21), (22) and (23), the exponent of  $\phi$  in unit time of  $z$  (I denote  $\mathcal{G}$ ) should be estimated by

$$\mathcal{G} = \int_{-\infty}^{\infty} d(\Delta W) P(\Delta W) \Theta(\mathcal{D}) \frac{\sqrt{\mathcal{D}}}{2}, \quad (27)$$

where  $\Theta(x)$  is the step function which is 1 for  $x > 0$  and 0 for  $x < 0$ , and  $\mathcal{D} = 4s^2 - 4\alpha(\Delta W)/\Delta z$ . This integration can be performed and I obtain the following expression of  $\mathcal{G}$ :

$$\mathcal{G} = \frac{\alpha^{1/2}}{2^{3/2}(\Delta z)^{1/4}} \exp \left( -\frac{(\Delta z) s^4}{4\alpha^2} \right) D_{-3/2} \left( -\frac{(\Delta z)^{1/2} s^2}{\alpha} \right), \quad (28)$$

where  $D_\nu$  is the parabolic cylinder function.<sup>27,28)</sup> I note that the expression  $\mathcal{G}$  depends on  $\Delta z$ . Nevertheless, I can read the behavior of the exponent as a function of  $\alpha$  from this expression. The quantity  $\mathcal{G}$  as a function of  $\alpha$  has an extremum which is determined by  $d\mathcal{G}/d\alpha$ . I use the following formulas of the parabolic cylinder function in order to simplify the expression of  $d\mathcal{G}/d\alpha$ :

$$\frac{d}{dx} D_p(x) - \frac{1}{2} x D_p(x) + D_{p+1}(x) = 0, \quad (29a)$$

$$D_{p+1}(x) - x D_p(x) + p D_{p-1}(x) = 0. \quad (29b)$$

As a result, I find that the sign of  $d\mathcal{G}/d\alpha$  depends on only the function  $D_{1/2}(x)$ . I obtain the subsequent condition that  $\mathcal{G}$  is extremum at a certain  $\alpha$ :

$$D_{1/2}\left(-\frac{(\Delta z)^{1/2}s^2}{\alpha}\right) = 0. \quad (30)$$

It is known that  $D_\nu(x)$  for positive  $\nu$  has  $[\nu + 1]$  zeros,<sup>29)</sup> where  $[\nu + 1]$  is the maximum integer which is not greater than  $(\nu + 1)$ . Then the equation,  $D_{1/2}(x) = 0$ , has one solution, and I write the solution as  $x_{\text{sol}}$ . This solution is negative, and then  $\alpha$  is positive at the extremum of  $\mathcal{G}$ . Therefore  $\mathcal{G}$  has one extremum surely at a positive  $\alpha$ . The value at the extremum of the exponent  $\mathcal{G}$  is given by

$$\mathcal{G}_{\min} = \frac{s}{2^{3/2}} \frac{1}{[-x_{\text{sol}}]^{1/2}} \exp\left(-\frac{1}{4}(x_{\text{sol}})^2\right) D_{-3/2}(x_{\text{sol}}). \quad (31)$$

It is shown that  $\mathcal{G}_{\min}$  is smaller than  $s$ , and then  $\mathcal{G}$  has one minimum. This indicates that the exponent  $\mathcal{G}$  is suppressed by white noise when the value of  $\alpha$  is adequate. I must note that the expression  $\mathcal{G}_{\min}$  is independent of  $\Delta z$ . Equation (28) for quite small  $s$  should be invalid, because the approximation of  $\Phi$  given in eq. (16) does not work well.

### 3. Numerical calculation of the exponent by a symplectic method

In this section, I attempt to solve eqs. (5a) and (5b) numerically. Our purpose is to investigate the amplification of the amplitude of  $\phi$  when white noise acts multiplicatively. Therefore, the amplitude must be calculated precisely, at least, when no periodic coefficient and no white noise term exist. It is well-known that the structure of such a system is symplectic. In the similar way, it was proved that the system has the symplectic structure even when noise exists if some conditions are satisfied.<sup>30)</sup> Taking this property into account, I use the symplectic method developed in ref. 31 in order to solve the stochastic differential equations with multiplicative white noise. The first-order method given in ref. 31 is applied to the equations in this paper. The initial conditions are  $\phi(0) = 1$  and  $\dot{\phi}(0) = 0$  in these calculations. The equations are solved numerically from  $z = 0$  to  $z = 500$ . The time step in  $z$  is set to 0.05.

In the case of  $\alpha \neq 0$ , one trajectory of  $\phi(z)$  can be calculated when the sequence of noise is given. I calculate many trajectories and take their average to obtain the mean value of the trajectories of the variable  $\phi_i^{(j)}(z)$ , where the subscript  $i$  indicates the batch and the superscript  $(j)$  indicates the trajectory in a certain batch  $i$ . In the present calculation, one batch contains 500 trajectories and 20 batches are taken. I calculate the mean value  $\mathcal{M}_i(z)$  of the trajectories in the batch  $i$ . The mean value over 20 batches,  $\bar{\phi}(z)$ , is given by

$$\bar{\phi}(z) = \frac{1}{20} \sum_{i=1}^{20} \mathcal{M}_i(z), \quad \mathcal{M}_i(z) = \frac{1}{500} \sum_{j=1}^{500} \phi_i^{(j)}(z). \quad (32)$$

I note that it is possible to perform interval estimation by using  $\bar{\phi}$  and  $\mathcal{M}_i$ . In the case of  $\alpha = 0$ , there

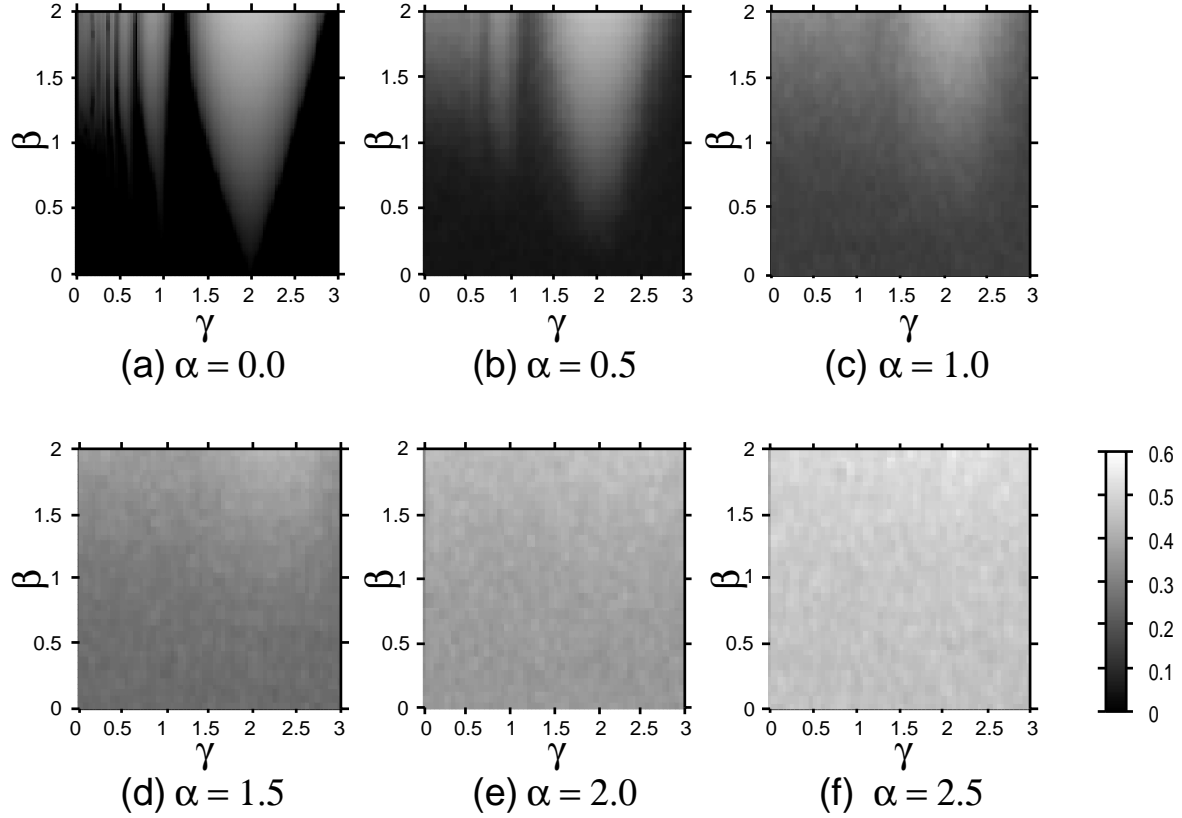


Fig. 1. Exponents on the  $\gamma$ - $\beta$  plane for various values of  $\alpha$ . The exponents are calculated by solving the stochastic differential equations numerically by the symplectic method. The parameters are (a) $\alpha = 0.0$ , (b) $\alpha = 0.5$ , (c) $\alpha = 1.0$ , (d) $\alpha = 1.5$ , (e) $\alpha = 2.0$ , (f) $\alpha = 2.5$  respectively.

is no need to calculate many trajectories. Therefore one trajectory is calculated numerically.

The exponent is estimated from the average  $\bar{\phi}(z)$  in the range of  $200 < z < 500$  in order to decrease the effects of the initial conditions. This estimation is performed as follows. 1) the set  $(z_k, \ln \bar{\phi}(z_k))$  is determined, where  $z_k$  is the time at which  $\bar{\phi}(z_k)$  is a local maximum and positive. 2) the set is fit with a linear function. The coefficient of the time  $z$  is adopted as the exponent. Here, I note the reason why the values,  $\ln \bar{\phi}(z_k)$ , are fit. One way to estimate the parameters is to fit the average  $\bar{\phi}(z_k)$  directly. In such the method, it is implicitly assumed that the dispersion of the distribution of the data at time  $z$  and that at time  $z' (\neq z)$  are the (approximately) same. However, the dispersion is wider with time  $z$  in the present case, because the treated process is a wiener process. The effects of non-equivalent dispersions are decreased by taking the logarithm of the data. Therefore the transformed data,  $\ln \bar{\phi}(z_k)$ , are fit with the linear function. I may notice that the exponents extracted by the above procedure are different generally from the Lyapunov exponents which are estimated by the mean value of the logarithm of  $\phi_i^{(j)}$ . The quantity,  $\ln \bar{\phi}(z_k)$ , is calculated, because I focus on the enhancement of the variable  $\phi$  in this paper.



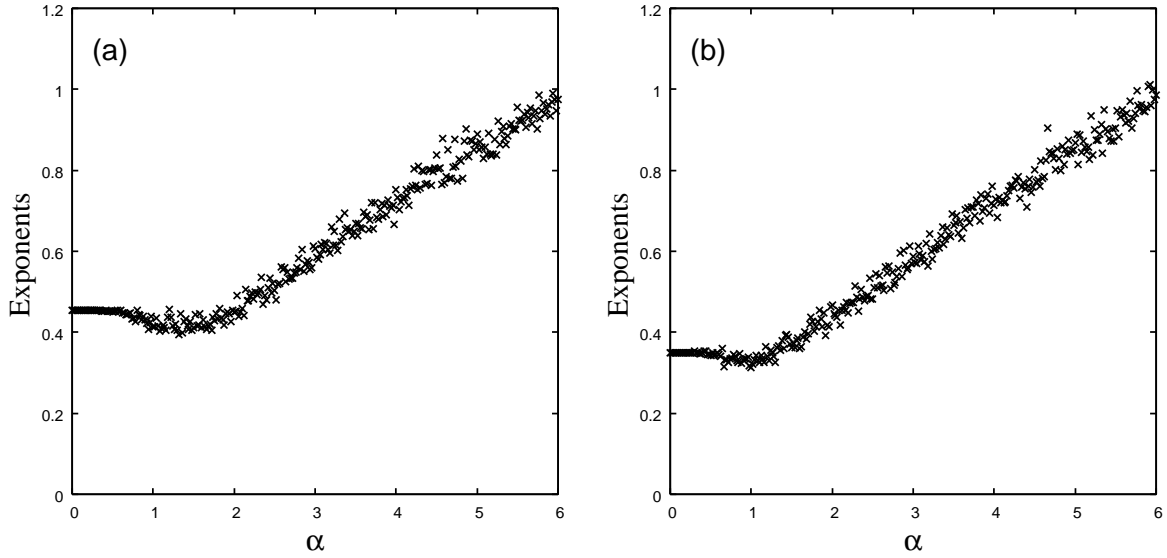


Fig. 2. Exponents on the resonance bands. The cross represents the data estimated from numerical results obtained by solving the eqs. (5a) and (5b). (a) The values of the parameters,  $\beta$  and  $\gamma$ , are both 2. (b) The values of the parameters,  $\beta$  and  $\gamma$ , are 2 and 0.9 respectively.

Figure 1(a) is the map of the exponents of Mathieu equation, eq. (7), on the  $\gamma$ - $\beta$  plane. The step sizes in  $\gamma$  and  $\beta$  in the numerical calculations are taken to be 0.02 to draw this figure. I denote these step sizes as  $\Delta\gamma$  and  $\Delta\beta$  respectively. The color of a square is determined from the arithmetic mean of the exponents at four corners which are located at  $(\gamma, \beta)$ ,  $(\gamma + \Delta\gamma, \beta)$ ,  $(\gamma, \beta + \Delta\beta)$  and  $(\gamma + \Delta\gamma, \beta + \Delta\beta)$ . The resonance band around  $\gamma = 2$  corresponds to the first resonance band of eq. (6). In the same way, the resonance band around  $\gamma = 1$  corresponds to the second resonance band. The  $n$ th resonance band of eq. (6) corresponds to the band around  $\gamma = 2/n$ , where  $n$  is positive integer.

Next, I show the map of the exponents for various values of  $\alpha$  on the  $\gamma$ - $\beta$  plane. Figure 1(b) is the map at  $\alpha = 0.5$ , (c) is at  $\alpha = 1.0$ , (d) is at  $\alpha = 1.5$ , (e) is at  $\alpha = 2.0$ , and (f) is at  $\alpha = 2.5$ . The step sizes in  $\beta$  and  $\gamma$  are 0.05 in the numerical calculations for Figs. 1(b), 1(c), 1(d), 1(e) and 1(f). The color of a square is determined in the same manner as in Fig. 1(a). The resonance band of the Mathieu equation is shown in Fig. 1(a). As shown in Figs. 1(b), 1(c), 1(d), 1(e) and 1(f), the band structure is destroyed by white noise, and the values of the exponents become large with  $\alpha$  for many sets of  $(\gamma, \beta)$ . However it seems from these figures that the exponent on the resonance band is not a monotonically increasing function of  $\alpha$ . Moreover, the  $\beta$  dependence of the exponent in Fig. 1(f) is weak as compared with those in other figures: Figs. 1(a), 1(b) and 1(c). This implies that the values of the exponents of the equation with the periodic coefficient are close to those without the periodic coefficient in Fig. 1(f). (The values of the exponents at  $\beta = 0$  correspond to the values in the case of no periodic coefficient.) It is evident

that the effects of the periodic coefficient become weak relatively.

Furthermore, I investigate the  $\alpha$  dependence of the exponent on the first and the second resonance bands. I draw the  $\alpha$  dependence of the exponent with fixed parameters,  $\gamma$  and  $\beta$ . I show the exponents for the set  $(\gamma = 2, \beta = 2)$  on the first resonance band, and the set  $(\gamma = 0.9, \beta = 2)$  on the second resonance band. Figure 2 shows the  $\alpha$  dependences of the exponents. The cross represents the data obtained by solving eqs. (5a) and (5b) numerically. The suppression by white noise is obviously seen and there is only one local minimum in each figure. The exponent decreases with  $\alpha$  and reaches the minimum. It continues to increase with  $\alpha$  after that. This behavior is interpreted as follows. The growth of the amplitude depends on the mechanism of parametric amplification for small  $\alpha$ . This mechanism is destroyed by noise with the increase of  $\alpha$ . Then the exponent decreases with  $\alpha$ . Contrarily, the amplitude is amplified by noise for large  $\alpha$ , as shown in many researches. In summary, the exponent decreases with  $\alpha$ , reaches the minimum, and increases after that. The exponents for other parameter sets,  $(\gamma, \beta)$ , on the resonance bands behave similarly.

Finally, the minimum value of the exponent as a function of  $\alpha$  is estimated for various values of  $\beta$  and  $\gamma$ . I denote the minimum value of the exponent estimated numerically as  $s_{\min}$ . Obviously  $s_{\min}$  is a function of  $\beta$  and  $\gamma$ . In this calculation, the range of  $\alpha$  is set to  $[0, 2]$  and the step size in  $\alpha$  is set to 0.01. The range of  $\gamma$  is set to  $[0.7, 2.7]$  and the step size in  $\gamma$  is set to 0.5. The exponents for various values of  $\alpha$  with fixed  $\beta$  and  $\gamma$  are estimated and  $s_{\min}$  is set to the minimum value of these exponents. I plot the quantity  $s_{\min}/s$ , because the exponent at  $\alpha = 0$ ,  $s$ , is also a function of  $\beta$  and  $\gamma$ . I show the values  $s_{\min}/s$  for  $s \geq 0.3$  in order to compare them with the value  $\mathcal{G}_{\min}/s$ . The value  $\mathcal{G}_{\min}/s$  is approximately 0.893 from eq. (31).

Figure 3 shows the values  $s_{\min}/s$  for  $s \geq 0.3$  and the exponents  $s$ . The parameter  $\beta$  is set to 2.0 in Fig. 3(a) and 1.5 in Fig. 3(b). Cross represents data points of  $s_{\min}/s$  and broken line indicates  $\mathcal{G}_{\min}/s$ . Asterisk represents data points of  $s$ . As seen in Fig. 2, noise influences the values in the numerical calculations. Therefore it is likely that the ratio  $s_{\min}/s$  fluctuates and that the values  $s_{\min}/s$  around the maximum of  $s$  are below the value  $\mathcal{G}_{\min}/s$ . Then I calculate also the simple moving average of the exponents as a function of  $\alpha$  with fixed  $\beta$  and  $\gamma$ , and attempt to find the minimum value of them. I take the average of  $n$  adjoining exponents and denote this average as  $s_{\min}^{\text{SMA}n}$ . For an example, the minimum of the averages of three adjoining exponents is represented as  $s_{\min}^{\text{SMA}3}$ . Figure 4 displays the  $s_{\min}^{\text{SMA}3}/s$  for  $s \geq 0.3$  and the exponents  $s$ . The parameter  $\beta$  is set to 2.0 in Fig. 4(a) and 1.5 in Fig. 4(b). The symbols in Figs. 4(a) and 4(b) are the same as in Figs. 3(a) and 3(b). It is found from Figs. 3 and 4 that  $\mathcal{G}_{\min}/s$  is close to the values estimated by numerical calculations around the peaks of  $s$  in the resonant regions.

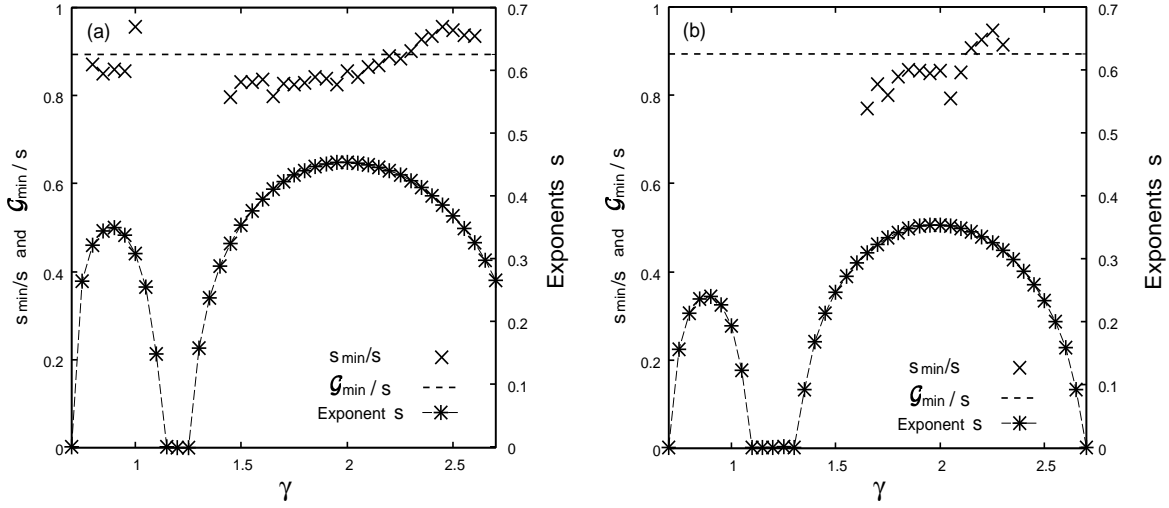


Fig. 3. The values  $s_{\min}/s$  for various values of  $\gamma$ . The parameter  $\beta$  is set to (a)  $\beta = 2.0$  and (b)  $\beta = 1.5$ . Cross represents the data points of  $s_{\min}/s$  and asterisk represents the data points of  $s$ . Broken line is the value  $G_{\min}/s$  that is approximately 0.893.

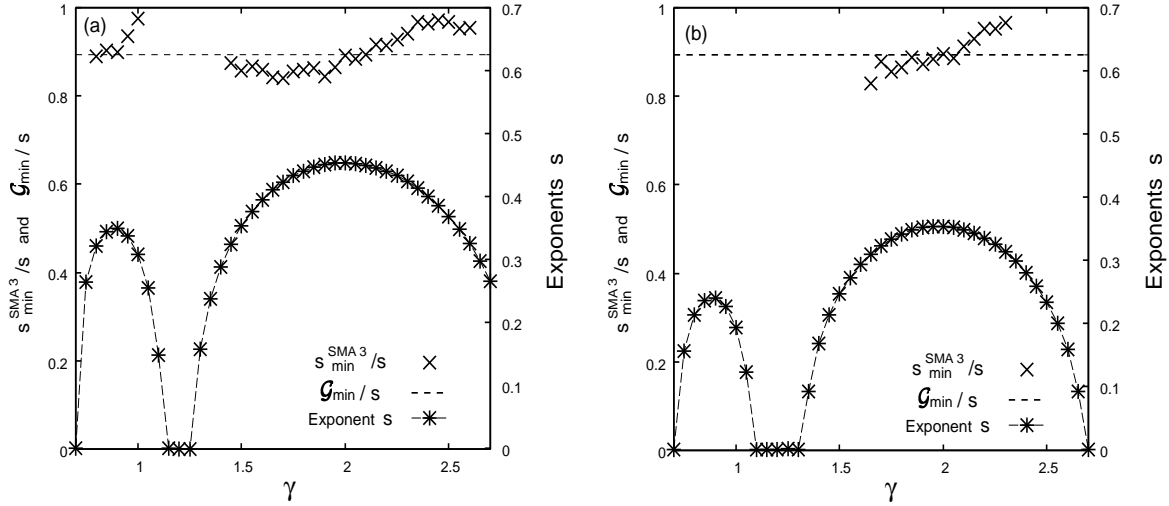


Fig. 4. The values  $s_{\min}^{\text{SMA3}}/s$  for various values of  $\gamma$ . The parameter  $\beta$  is set to (a)  $\beta = 2.0$  and (b)  $\beta = 1.5$ . Cross represents the data points of  $s_{\min}/s$  and asterisk represents the data points of  $s$ . Broken line is the value  $G_{\min}/s$  that is approximately 0.893.

#### 4. Discussion and Conclusion

I studied the amplification in a Mathieu-like equation in the presence of multiplicative white noise. An approximate expression of the exponent was derived by introducing the width of time interval on parametrically amplified regions where parametric amplification occurs when no noise exists. The exponents were calculated by solving the stochastic differential equations numerically by the symplectic

numerical method. The intensity of noise and the strength of the coupling between the noise and the variable are reflected to the value of the parameter  $\alpha$ . The behavior of the exponents as a function of  $\alpha$  is extracted.

With regard to the effects of multiplicative white noise on the amplification, it was shown that the band structure of the Mathieu equation is destroyed when white noise exists. The resonance structure survives for small values of  $\alpha$ , and this structure is lost for large values of  $\alpha$ . In the previous paper,<sup>32)</sup> I investigated the amplification in a stochastic differential equation without a periodic coefficient, and found that the exponent is a monotone increasing function of  $\alpha$ . In contrast, it is found in the present study that the exponent as a function of  $\alpha$  has one minimum on the parametrically amplified region of  $\alpha = 0$ . This indicates the suppression of the amplification by white noise which occurs when the value of  $\alpha$  is adequate. Equation (28) can roughly explain the behavior of the exponent as a function of  $\alpha$ : The exponent decreases with  $\alpha$ , reaches the minimum and increases after that.

One expects that the exponent as a function of the intensity of white noise has one minimum intuitively. However the exponent may have some minima caused by white noise. Theoretical expression given by eq. (30) with the cylinder function indicates that only one minimum exists. This fact is numerically supported too. It is shown in the previous sections that the exponent as a function of  $\alpha$  has one minimum theoretically and numerically.

The minimum value of the exponent as a function of  $\alpha$  is estimated from the numerical calculations. I calculated the ratio  $s_{\min}/s$ : the minimum value divided by the exponent at  $\alpha = 0$ ,  $s$ . This ratio obtained numerically is in rough agreement with that obtained theoretically around the peaks of  $s$  on the resonant regions. The minimum value of the exponent is approximately proportional to the exponent  $s$ . The relative variation is of the order of 90%, as shown in the figures and eq. (31). It seems that the variation is small. Nevertheless, the amplitude of the observable is affected strongly after long time passes, because this is the variation of the exponent.

The decrease of Lyapunov exponent by noise was found in the system of an inverted Duffing oscillator with noise.<sup>33)</sup> The mechanism of the amplification in the present case is different from that in the case of the inverted Duffing oscillator, when noise is absent. However, the mechanism of the suppression in the present case is surely the same as that in the case of the inverted Duffing oscillator. In both cases, the amplification is suppressed by noise which intensity is adequate. The exponent decreases with the intensity of noise, and reaches the minimum. After that, the exponent increases with the intensity of noise. The decrease of the exponent by white noise implies the possibility of the large decrease by colored noise, though the suppression by white noise is weak in the present case. Then the system that parametric resonance occurs may be stabilized by colored noise, as found in the system of the inverted Duffing oscillator.

The expression of the exponent obtained theoretically includes the artificial parameter  $\Delta z$ . Then the value of  $\alpha$  at the minimum of  $\mathcal{G}$  depends on  $\Delta z$ , while the minimum value of  $\mathcal{G}$  is independent of  $\Delta z$ . The value of  $\mathcal{G}$  for large  $\alpha$  are out of accord with that obtained theoretically, because the expression of  $\mathcal{G}$  is calculated in the case that the exponent  $s$  is important. These differences may come from the fact that the effects of white noise on parametric amplification are not reflected sufficiently. I would like to solve these problems in the future study.

### **Appendix: The effects of white noise on the exponent of the stochastic differential equation with a periodic coefficient**

In this paper, I treated the stochastic differential equation with a periodic coefficient which varies sinusoidally. In the same way, I can discuss a stochastic differential equation with coefficients varying periodically. I treat the following equation:

$$\ddot{\phi}(z) + [h(z) + \alpha r(z)] \phi(z) = 0, \quad (\text{A}\cdot 1)$$

where  $\alpha$  is constant,  $r(z)$  is white noise which properties are given in eq. (4) and the dot represents the derivative with respect to  $z$ . The function  $h(z)$  is a periodic function of period  $\pi$ .

I treat the subsequent equation to analyze the solution of eq. (A·1):

$$\ddot{\Phi}(z) + h(z)\Phi(z) = 0. \quad (\text{A}\cdot 2)$$

From Floquet's theorem, this differential equation has a solution as follows:

$$\Phi(z) = e^{i\mu z} F(z), \quad F(z) = F(z + \pi). \quad (\text{A}\cdot 3)$$

Substituting  $\phi(z) = \Phi(z)\psi(z)$  into equation (A·1), I obtain

$$\ddot{\psi} + 2(\dot{\Phi}/\Phi)\dot{\psi} + \alpha r(z)\psi = 0. \quad (\text{A}\cdot 4)$$

By using eq. (A·3), eq. (A·4) is given by

$$\ddot{\psi} + 2(i\mu + \dot{F}/F)\dot{\psi} + \alpha r(z)\psi = 0. \quad (\text{A}\cdot 5)$$

As shown in §2.2, I obtain an approximate differential equation for  $\psi$  when the average of the coefficient of  $\dot{\psi}$  in the period  $\pi$  is taken:

$$\ddot{\psi} + 2i\mu\dot{\psi} + \alpha r(z)\psi = 0. \quad (\text{A}\cdot 6)$$

I treat the case that  $\mu$  is pure imaginary, because I study the case that the amplitude of  $\Phi$  grows. It is conjectured that the effects of the constant coefficient of  $\dot{\psi}$ ,  $2i\mu$ , on the growth of  $\psi$  is stronger than those of the periodic coefficient of  $\dot{\psi}$ ,  $2\dot{F}/F$ .

Setting  $\mu$  as  $-is$  where  $s$  is positive constant, eq. (A·6) is rewritten as

$$\ddot{\psi} + 2s\dot{\psi} + \alpha r(z)\psi = 0. \quad (\text{A} \cdot 7)$$

This equation is just eq. (18). Then the final expression of the exponent of  $\phi(z)$  is equal to eq. (28). Therefore the exponent as a function of  $\alpha$  has one minimum on the region where  $\Phi$  grows exponentially. This result suggests that the amplification of the variable  $\phi(z)$  that obeys the stochastic differential equation with the periodic coefficient, eq. (A·1), is suppressed by white noise when the variable  $\Phi(z)$  grows and the parameter  $\alpha$  is adequate, as shown in §2.2 and 3.

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